tion was 2.26, or 0.54 below normal; the greatest monthly amount, 3.45, occurred at East Bank, and the least, 1.22, at Beckley.

Early sown wheat is reported in excellent condition generally over the State, except in the extreme southern portion, where some damage has been done by the fly. Late sown wheat has been injuriously affected by the freezing and thawing, so that it has not made much progress. Some plowing has been done for corn and stock is looking fairly well.—*E. C. Vose*

Wisconsin.—The mean temperature was 23.5°, or 1.6° above normal; the highest was 52°, at Brodhead on the 22d, and the lowest, 21° below zero, at Grantsburg on the 31st. The average precipitation was 0.71, or 0.88 below normal; the greatest monthly amount, 2.11, occurred at Casco, and the least, 0.07, at Delavan.—*W. M. Wilson.*

Wyoming.—The mean temperature was 28.8°, or 4.9° above normal; the highest was 68°, at Fort Washakie on the 7th, and the lowest, 34° below zero, at same station on the 31st. The average precipitation was 0.44, or 0.21 below normal; the greatest monthly amount, 1.18, occurred at Fort Yellowstone, while none fell at Hyattville.—*W. S. Palmer.*

SPECIAL CONTRIBUTIONS.

THE CIRCULATORY MOVEMENTS IN THE ATMOSPHERE.

By Prof. V. BJERKNES.

In the MONTHLY WEATHER REVIEW for October, 1900, pp. 434–443, we have given the complete translation of a memoir by Prof. V. Bjerknes in which he explains in the most elementary manner his excellent ideas as to the geometrical treatment of a general dynamic principle applicable to the movements of the atmosphere. Possibly some of our readers will appreciate the further elucidation of this subject that has just been published by Bjerknes in reply to criticisms by a European student. We have, therefore, prepared the following collection of extracts from Bjerknes' article in the Meteorologische Zeitschrift for November, 1900, pp. 481–491, omitting some matters that are not essential to the proper understanding of his explanations.—ED.

The term gradient is now applied in meteorology to a series of quantities of very various physical significations. But all these quantities have certain common mathematical peculiarities corresponding to the meaning of this word. The gradients are all *directed* quantities, or vector quantities, whose distribution in space can be described by means of a system of surfaces. Along each surface of this system of surfaces a certain nondirected or scalar quantity has a constant value, and corresponding to this property we designate these surfaces by an additional word that is formed from the name of the scalar quantity itself with the prefix iso- or equi-[1]. The gradient is everywhere directed perpendicularly to these surfaces, and shows the direction and the amount of the greatest rate of change in that scalar quantity, which is constant in the direction of the (iso-) or (equi-) surfaces.

It is customary to give the corresponding gradient the same name as that of the scalar quantity. As a typical example, we may consider the temperature. This scalar quantity is constant along the isothermal surfaces. The temperature gradient or the thermal gradient is directed perpendicular to these surfaces and shows the direction and the amount of the greatest fall of temperature.

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I request the readers of my previous memoir to recall that I have used the word gradient only in the sense of the barometric gradient.

For the sake of greater clearness and in order to deduce some general properties of these vector quantities, I will write out the general hydrodynamic equations of motion. If x, y, z are the coordinates of any given particle of fluid, U_x, U_y, U_z are the component velocities, and, consequently,

$$\frac{dU_x}{dt}, \frac{dU_y}{dt}, \frac{dU_z}{dt}$$

are the component accelerations corresponding to these, then the equations of motion can be written in the following simple form, where p is the pressure, q the density, and g, with its components g_x, g_y, g_z, is the exterior accelerating force acting upon each unit of mass of the fluid:

$$(1)\qquad \begin{aligned} q\frac{dU_x}{dt} &= -\frac{\partial p}{\partial x} + qg_x \\ q\frac{dU_y}{dt} &= -\frac{\partial p}{\partial y} + qg_y \\ q\frac{dU_z}{dt} &= -\frac{\partial p}{\partial z} + qg_z. \end{aligned}$$

In the case of atmospheric motions, g is the acceleration of gravity and qg the weight of a unit volume of air. On the right-hand side of these equations there also occur those components of the vector quantity about whose proper designation we have spoken, viz, the barometric gradient G, whose components are

$$(2)\qquad \begin{aligned} G_x &= -\frac{\partial p}{\partial x} \\ G_y &= -\frac{\partial p}{\partial y} \\ G_z &= -\frac{\partial p}{\partial z}. \end{aligned}$$

On the other hand, by the space gradient Gr of Möller we must understand a vector quantity whose components along the axes are completely expressed by the right-hand members of the following equations:

$$(3)\qquad \begin{aligned} Gr_x &= -\frac{\partial p}{\partial x} + qg_x \\ Gr_y &= -\frac{\partial p}{\partial y} + qg_y \\ Gr_z &= -\frac{\partial p}{\partial z} + qg_z. \end{aligned}$$

* * * * * * *

The hydrodynamic equations (1) form the starting point of my study. I did not write them out in my previous memoir because I wished to give the demonstration the simplest possible elementary form; but, essentially, these equations did form my starting point, and since the right-hand sides of these equations are the three components of Dr. Möller's "space gradient Gr," therefore I have taken complete account of his vector quantity. It is no error, but rather a very important advantage that this vector does not appear in my result, for the vector Gr is in most cases a very awkward quantity to handle, and it would be difficult to make much advance by using it. In order to make this perfectly plain, I will repeat this portion of the proof in a purely analytical form. I first rearrange the equations (1), in that I divide throughout by the density, q, and then replace the density by its reciprocal, viz, the specific volume, or $k = 1/q$. Thus the equations become

$$(4)\qquad \begin{aligned} \frac{dU_x}{dt} &= -k\frac{\partial p}{\partial x} + g_x \\ \frac{dU_y}{dt} &= -k\frac{\partial p}{\partial y} + g_y \\ \frac{dU_z}{dt} &= -k\frac{\partial p}{\partial z} + g_z. \end{aligned}$$

[1] According to the terminology introduced by Hamilton, and now generally used in mathematical physics, a quantity is said to be *scalar* when its value, at any point in space, can be expressed by a single number; typical examples are density, pressure, temperature, relative humidity, potential, etc. On the other hand, a quantity is said to be a *vector* quantity when three numbers are necessary in order to specify its value at any given point in space. The vector quantities have both magnitude and direction, the three numbers referred to are their components; typical examples are velocity, acceleration, force, and all quantities that we call gradients.

On the left-hand side of the equations in this form are the components of the acceleration, and on the right-hand side, according to the terminology that has been used ever since the days of Sir Isaac Newton, we have the total accelerating force. This total accelerating force is the sum of two simpler accelerating forces, viz, the accelerating gravity, g, also known more simply as the acceleration of gravity, and again the accelerating force due to the barometric gradient, whose components along the respective axes are:

$$-k\frac{\partial p}{\partial x}, -k\frac{\partial p}{\partial y}, -k\frac{\partial p}{\partial z}.$$

Now let ds be the line element of a curve consisting of particles of the fluid, and dx, dy, dz, be the projections of this element on the coordinate axes. Multiply the equations (4) respectively by dx, dy, dz, add the products and integrate along the whole curve under the assumption that it is a closed curve; there results the following:

$$\begin{aligned}(5)\qquad &\int\left\{\frac{dU_x}{dt}dx+\frac{dU_y}{dt}dy+\frac{dU_z}{dt}dz\right\}=\\ &=-\int k\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right)\\ &+\int\left(g_x\,dx+g_y\,dy+g_z\,dz\right)\end{aligned}$$

The integral on the left hand, under the special assumption that the curve is closed can be written in the form

$$(6)\qquad \frac{d}{dt}\int(U_x dx+U_y dy+U_z dz).$$

This transformation is given in Lord Kelvin's original proof of his theorem as to the conservation of circulatory motion,[2] and is also, in a more or less complete form, to be found in all text-books that teach this theory.[3] Moreover, the same transformation is given in the developments on Pages 100 and 101 of my memoir.[4] The quantity following the integral sign in equation (6) is simply the circulation, C, of the curve, according to the terminology introduced by Lord Kelvin. Therefore, the left-hand side of equation (5) reduces to $\frac{dC}{dt}$, which is the increase in the circulation of the curve within a unit of time.

The first integral on the right-hand side of equation (5) also has a simple meaning. The expression within the parenthesis, viz:

$$\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz$$

can also be written

$$\frac{\partial p}{\partial s}ds.$$

But $-\frac{\partial p}{\partial s}$ is nothing else than the component of the barometric gradient, G, that is tangential to the curve. If we designate this tangential component by G_t then the integral can be written simply as

$$(7)\qquad \int kG_t\,ds.$$

Finally, the second integral on the right-hand side of equation (5) disappears identically along the closed curve, if g represents the acceleration of gravity; for every force for which this integral does not disappear will, as is well known, maintain a perpetual motion.

[2] Sir William Thomson On Vortex Motion, Trans. Roy. Soc., Edinburgh, 1869, Vol. XXV, p. 217.

[3] See, for example, Poincaré's Théorie des Tourbillons, p. 12, Paris, 1893.

[4] See Met. Zeitschr., March, 1900, or MONTHLY WEATHER REVIEW, October, 1900, p. 435.

Therefore, finally, equation (5) reduces to the following:

$$(8)\qquad \frac{dC}{dt}=\int k\,G_t\,ds$$

If we recall that kG represents the accelerating force arising from the barometric gradient, and, further, that by the line-integral of a vector quantity we always understand the line-integral of the component tangential to the curve, then the equation (8) can be expressed in words as follows: *The increase of circulation in a unit of time of a closed curve is equal to the line-integral of that accelerating force which arises from the barometric gradient.*

This theorem is identical with that given in my previous memoir,[5] viz, that the increase of circulation is equal to the number of the solenoids inclosed within the curve. In fact, the number of the solenoids, A, is simply another expression for the above integral (No. 7).

* * * * * * *

It will be noticed that in passing from equation (1) to the equations (4), we no longer have to do with the quantity Gr, or the "space gradient" of Möller, as such, but with its product into the specific volume. In the integration along the closed curve, the gravity term drops out in consequence of the conservative nature of this force, so that only the barometric gradient multiplied by the specific volume remains under the integral sign.

Of course this does not mean that the gravitating force has no importance in the circulation of the air, but only that, in the method of computation here applied, this force does not enter explicitly into the formula. As seen from the point of view here chosen, its action is indirect; it exerts a joint influence on the distribution of pressure and density, and its influence is, therefore, indirectly included in the integral.[6] As will be shown by Sandström in a memoir soon to be published, this influence of the gravitating force may also be made to stand forth explicitly. But it appears that invariably, in the computation of the circulation, the knowledge of either the barometric gradient or the gravitating force, alone, in connection with the density or the specific volume of the air, will suffice, whereas we never need to know the complete "space gradient, Gr."

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If a vector quantity is given in space, we can always draw curves that shall be everywhere tangent to the direction of the vector; but, as is known to every geometer, it is only exceptionally possible to construct curved surfaces that shall be everywhere perpendicular to those curves, and, therefore, also to the vector quantities themselves. If we indicate any vector quantity by U and its components along the axes by U_x, U_y, U_z, then a surface normal to the vector quantity, U, can only be constructed when the three quantities U_x, U_y, U_z, satisfy the following equation:

$$(9)\quad U_x\left(\frac{\partial U_z}{\partial y}-\frac{\partial U_y}{\partial z}\right)+U_y\left(\frac{\partial U_x}{\partial z}-\frac{\partial U_z}{\partial x}\right)+U_z\left(\frac{\partial U_y}{\partial x}-\frac{\partial U_x}{\partial y}\right)=0$$

We have here to do with a general principle long since known to geometers, but which one in his haste, easily overlooks. For it seems easy to say that we may draw a surface element normal to each individual curve of a system of curves, and that the totality of all these elementary surfaces must constitute a continuous system of surfaces perpendicular to the original curves; but the incompleteness of this statement is readily recognized when we consider a simple example. For instance, let the curves resemble the fibers of an ordinary twisted thread. If we attempt to imagine the nature of the

[5] Met. Zeitschr., p. 106, or MONTHLY WEATHER REVIEW, pp. 437-8.

[6] The attention of the reader is called to the fact that in fig. 8, Met. Zeitschr., 149, or MONTHLY WEATHER REVIEW, p. 439, the vertical arrow on the right-hand side should point downward, not upward, as in the figure.

surface we see at once that it is impossible to draw any surface perpendicular to curves that have such courses as these. If we twist the fiber backward so that it is made up of parallel straight lines then we can draw a plane surface perpendicular thereto. For different degrees of bending or deformation of such a bundle of fibers, this plane will be curved and deformed, but still without ceasing to constitute a connected surface. But the instant that we twist the bundle of fibers the plane will break into individual elementary surfaces that can no longer be combined into a continuous surface.

Now, equation (9) expresses the mathematical condition that the curves running tangential to the vector quantities shall show no such torsion as those of the fibers of a thread. In order to investigate the relation of the vector quantity, Gr, in reference to this property, we may insert in equation (9) the values of the components as given in equation (3). After a simple reduction there results the following:

$$(10)\qquad \frac{\partial p}{\partial x}\left(g_z\frac{\partial q}{\partial y}-g_y\frac{\partial q}{\partial z}\right)+\frac{\partial p}{\partial y}\left(g_x\frac{\partial q}{\partial z}-g_z\frac{\partial q}{\partial x}\right)+\frac{\partial p}{\partial z}\left(g_y\frac{\partial q}{\partial x}-g_x\frac{\partial q}{\partial y}\right)=0$$

The discussion of this equation, according to regular geometrical methods, leads to the following results: it will be satisfied, and then only, when a system of curves can be given along which, and simultaneously, three scalar quantities are constant, viz: density, pressure, and the potential of gravity. These curves must, necessarily, lie in the level surfaces of gravity, and the condition that must be satisfied may therefore be formulated as follows: The equation (10) will be satisfied when in every level surface of gravity a system of curves can be drawn such that along every curve the value both of the density and of the pressure is constant. This condition, as may easily be seen, can be fulfilled in a series of special cases, which are completely enumerated in the following list:

1. When the pressure is constant throughout the whole space.
2. When the density is constant throughout the whole space.
3. When the surfaces of equal pressure coincide with the level surfaces of gravity.
4. When the surfaces of equal density coincide with the level surfaces of gravity.
5. When the surfaces of equal pressure and those of equal density coincide with each other.
6. When the lines of intersection of the surfaces of equal pressure and equal density fall in the level surfaces of gravity.

The first case is at present of no importance. The second case is important, since this condition is always fulfilled when the fluid is homogeneous and incompressible. The distribution of the moving forces in such a fluid can, therefore, be expressed by a diagram of isostenes, or isostenic surfaces [which term is applied by Möller to surfaces that are perpendicular to the space gradient or the vector quantity, Gr]. If p is the pressure and φ the potential of gravity, then the equation of these isostenic surfaces is

$$\varphi - p = \text{constant},$$

and in the solution of special problems these will often find application in the literature of hydrodynamics. In this case the vector, Gr, is a quantity of the nature of a gradient. The conditions (3) and (4) can be momentarily fulfilled in the course of any movement, but only to be not fulfilled in the next moment. In such isolated moments, therefore, isostenic surfaces can be drawn, but only to disappear in the next instant. These conditions can be permanently fulfilled only in the case of equilibrium. On the other hand, case (5) is important because the condition that the surfaces of equal density and equal pressure shall agree will always be fulfilled if we assume that the density is a function of the pressure only. In this case, also, there exist isostenic surfaces whose equation can, in general, be written as

$$\varphi - \int\frac{dp}{q} = \text{constant},$$

and these will find frequent application in the solution of special problems. All movements of the type of sound waves also belong to the category of atmospheric motions that we may study under the assumption that the density of the air depends only on the pressure. In this category, also, belongs the concrete example of the courses of the isostenic surfaces that was given by Möller.[7] Finally, case (6) refers to fluid media of the most general nature, and the condition now in question is again such an one as can only be fulfilled accidentally, for an instant, and in the next instant will in general cease to be fulfilled.

Therefore, the isostenic surfaces will, in general, be useful only in cases (2) and (5) that is to say, in homogeneous, incompressible fluids and in fluids where the density is a function of the pressure only. But it is precisely to these two cases that the celebrated Helmholtz-Kelvin theorem relates, according to which a circulatory motion can not be initiated, and, inversely, a circulatory motion once established also can not be annihilated. When we make these special assumptions, as to the property of the fluid, we thereby exclude the possibility of discussing circulatory motions in the atmosphere.

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Finally, I return to the terminology, which we meet with in dynamic meteorology, of the directed quantities distributed throughout all space.

In dynamic meteorology we meet first with two vector quantities distributed throughout the whole space and which belong to the simplest category of vector quantities. These are: (1) The force of gravity acting on a *unit of mass*, generally called the acceleration of gravity, and (2) the moving force resulting from pressure acting on the *unit volume*, or the quantity that I have here called the barometric gradient. These vector quantities have not only the property that one may draw surfaces perpendicular to them, viz, the level surfaces of gravity, or the isobaric surfaces of pressure, but we may also, by means of these surfaces, express the magnitude of the vectors, since the latter are inversely proportional to the thickness of the layer between two consecutive surfaces. Therefore, in accordance with Lord Kelvin's notation, these two vector quantities belong to the category of lamellar vector quantities. The conception of a lamellar vector is in mathematical sense identical with the conception of a gradient.

Next after these two simplest vector quantities, we meet two of a somewhat more complex nature, namely, the product of the density of the air by the acceleration of gravity and, again, the product of the specific volume of the air by the barometric gradient. The first of these quantities is the force of gravity acting on a unit of volume or the weight of a unit volume of air, understanding by this weight a directed quantity distributed throughout the whole space. The second of these quantities is the force resulting from the pressure acting upon a unit of mass. This is the vector quantity that we encountered in developing the theorem above given, as expressed in equation (3), and represents the accelerating force corresponding to the gradient of pressure. These two

[7] Met. Zeitschr., 1895, p. 92.

vector quantities have the further property of standing perpendicular to a system of surfaces, viz, the system of level surfaces, or isobaric surfaces, respectively. But the magnitudes of these quantities can no longer be expressed by the thickness of the lamella between the successive surfaces, but for this purpose we must introduce a second system of surfaces, viz, the surfaces of equal density, or equal specific volume. These two vector quantities therefore belong to the category of complex lamellar vector quantities that Lord Kelvin has so called on account of the two systems of surfaces and lamellæ. Therefore, in the strict sense of the word, these quantities are not of the nature of gradients. The importance of these vector quantities depends especially upon the fact that they are the simplest quantities upon which we can base the computation of the circulatory motions of the atmosphere. The accelerating barometric gradient is the vector quantity whose tangential component appears under the sign of integration in equation (8), and the tangential component of the weight of a unit volume of air will occur under the integration sign in the corresponding integral when we describe the circulation by the use of the specific moving quantities, viz, the product of velocity and density, instead of by velocity alone.

By the combination of a simple lamellar vector quantity (the barometric gradient) and a complex lamellar vector (the weight of the air in a unit of volume) there arises that vector quantity which Möller has called "the space gradient," *Gr*.

It therefore seems to me important to earnestly warn against the use of the term *gradient* for this vector quantity. Since, misled by the name gradient, we are liable to attribute to it the properties of a gradient, and be led into further error with respect to the isostenic surfaces. Moreover, a special name for this quantity is entirely unnecessary, for in mathematical relations it is nothing else than a vector quantity, and in mechanical aspects it is a force of a most general nature. The retention of the name gradient would also be equivalent to saying that in meteorology we call that a gradient which in mechanics is called force, and that in meteorology we speak of magnitudes of the nature of the gradient, when the mathematician speaks of vector quantities. This special name can be useful only when we apply the term gradient in a mechanical sense to forces of a very special nature, and in a mathematical sense to vectors of a very special nature, and then alone would the name have a prospect of being accepted from meteorology into the mathematical and mechanical sciences, and of assisting instead of hindering the cooperation of these sciences in meteorological questions.

The rational introduction of a terminology will, therefore, encounter no serious difficulty where the term gradient is used only in the above-mentioned special sense; but so far as I know no other meteorologist has accepted and used this special gradient. Only in one point do we come into disagreement with the old usage, viz, by the term vertical gradient we generally mean, not the vertical gradient of pressure, but the difference between the vertical gradient of pressure and the force of gravity. This is the vertical component of Möller's gradient. So long as we retain this term, instead of speaking of the vertical force, it will be very easy to call the general force directed at random in space, the *space* gradient, and misled by this name to attribute to this force the properties of the gradients and thereby again find ourselves tending toward the errors with regard to isostenic surfaces. Therefore, it would seem best not to make any further use of the term vertical gradient in the above sense, which is in fact not generally done, but to indicate the difference between the two forces by the simple term "vertical force."

73——2

LINE INTEGRALS IN THE ATMOSPHERE.

By FRANK H. BIGELOW.

The papers by Prof. V. Bjerknes[1], in connection with a criticism thereon[2] by Dr. M. Möller, and a practical application of the theory to a cyclone, by Dr. J. W. Sandström[3], have brought before meteorologists a problem of interest and practical importance. It is, therefore, desirable to investigate the bearings of the theory from various points of view. The following contribution to the subject is not intended as a criticism of the preceding works, but as a supplement to the discussion of the subject given by Prof. V. Bjerknes himself.[4]

What the theory is may readily be described in the following manner: The well known diagram by Hertz of the adiabatic changes in the condition of moist air, Abbe's translations, On the Mechanics of the Earth's Atmosphere, shows the relations between pressure B, temperature t, and vapor tension e, in the four stages, α, β, γ, δ. Now, since the density ρ is a function of B, t, e, only, a similar diagram will result for the function ρ, the density, and hence for the specific volume $v = \frac{1}{\rho}$, by drawing other lines on the same coordinate axes.

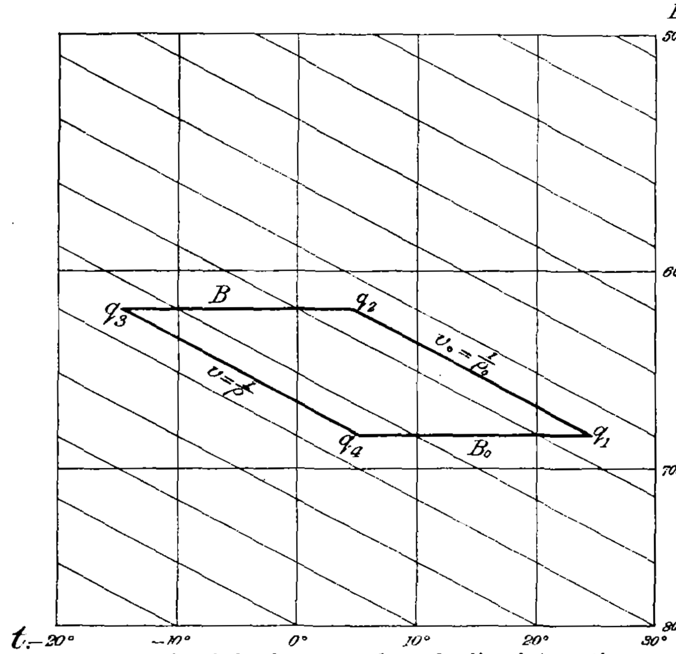

FIG. 1.—Circuit in the atmosphere for line integration.

Using throughout this paper the notation laid down in my International Cloud Observations Report, on pages 485–488, and also the formulæ derived in the sections following, we have

$$\frac{1}{\rho} = \frac{P_o(1+\alpha t)}{\rho_o P} \text{ for dry air,} \qquad 47a;$$

$$\text{and } \frac{1}{\rho} = \frac{1}{\rho_o}\frac{B_o}{B}(1+\alpha t)(1+\beta)(1+\gamma)\left(1+\frac{h+h_o}{R}\right).$$

$$= \frac{1}{\rho_o}\frac{760}{(B-.377e)}\cdot\frac{1+\alpha t}{n_1}, \text{ for moist air.}$$

[1] Das dynamische Princip. der Circulations bewegungen in der Atmosphäre. Meteorol Zeit. Marz 1900 und April 1900. Translation. WEATHER REVIEW, October, 1900.

[2] Der räumliche Gradient, M. Möller. Meteorol Zeit. June, 1900.

[3] Ueber die Verwendung von Prof. V. Bjerknes' Theorie der Wirbelbewegungen in Gasen und Flüssigkeiten. Königl. Schwed. Ak. der Wiss. January, 1900.

[4] Räumlicher Gradient und Circulation. Von V. Bjerknes. Meteorol. Zeits. November, 1900.